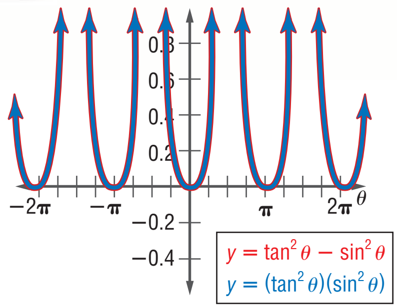
14-4 Verifying Trigonometric Identities

Main Ideas

- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

GET READY for the Lesson

Examine the graphs at the right. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of θ and conclude that the statement is true for all values of θ. In order to show that the equation $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ for all values of θ, you must consider the general case.



Transform One Side of an Equation You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity, you need to show that it is true for all values of θ.

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation *separately* until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

Study Tip

Common Misconception
You cannot perform operations to the quantities from each side of an unverified identity as you do with equations. Until an identity is verified it is not considered an equation, so the properties of equality do not apply.

EXAMPLE Transform One Side of an Equation

1 **Verify that $\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta$ is an identity.**

Transform the left side.

$$\tan^2 \theta - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Original equation}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Rewrite using the LCD, } \cos^2 \theta.$$

$$\frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Subtract.}$$

$$\frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} \stackrel{?}{=} \tan^2 \theta \sin^2 \theta \qquad \text{Factor.}$$

$$\frac{\sin^2\theta\sin^2\theta}{\cos^2\theta} \stackrel{?}{=} \tan^2\theta\sin^2\theta \qquad 1 - \cos^2\theta = \sin^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{\sin^2\theta}{1} \stackrel{?}{=} \tan^2\theta\sin^2\theta \qquad \frac{ab}{c} = \frac{a}{c}\cdot\frac{b}{1}$$

$$\tan^2\theta\sin^2\theta = \tan^2\theta\sin^2\theta \qquad \frac{\sin^2\theta}{\cos^2\theta} = \tan\theta$$

CHECK Your Progress

1. Verify that $\cot^2\theta - \cos^2\theta = \cot^2\theta\cos^2\theta$ is an identity.

STANDARDIZED TEST EXAMPLE — Find an Equivalent Expression

2 $\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\cot\theta}\right) =$

A $\cos\theta$ **B** $\sin\theta$ **C** $\cos^2\theta$ **D** $\sin^2\theta$

Read the Test Item

Find an expression that is equal to the given expression.

Solve the Test Item

Transform the given expression to match one of the choices.

$$\sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\cot\theta}\right) = \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}}\right) \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$= \sin\theta\left(\frac{1}{\sin\theta} - \frac{\cos\theta\sin\theta}{\cos\theta}\right) \qquad \text{Simplify.}$$

$$= \sin\theta\left(\frac{1}{\sin\theta} - \sin\theta\right) \qquad \text{Simplify.}$$

$$= 1 - \sin^2\theta \qquad \text{Distributive Property}$$

$$= \cos^2\theta$$

The answer is C.

Test-Taking Tip

Verify your answer by choosing values for θ. Then evaluate the original expression and compare to your answer choice.

CHECK Your Progress

2. $\tan^2\theta(\cot^2\theta - \cos^2\theta) =$

F $\cot^2\theta$ **G** $\tan^2\theta$ **H** $\cos^2\theta$ **J** $\sin^2\theta$

Transform Both Sides of an Equation Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.

EXAMPLE 3 Verify by Transforming Both Sides

Verify that $\sec^2 \theta - \tan^2 \theta = \tan \theta \cot \theta$ is an identity.

$\sec^2 \theta - \tan^2 \theta \stackrel{?}{=} \tan \theta \cot \theta$ — Original equation

$\frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta}$ — Express all terms using sine and cosine.

$1 - \frac{\sin^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$ — Subtract on the left. Multiply on the right.

$\frac{\cos^2 \theta}{\cos^2 \theta} \stackrel{?}{=} 1$ — $1 - \sin^2 \theta = \cos^2 \theta$

$1 = 1$ — Simplify the left side.

CHECK Your Progress

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

CHECK Your Understanding

Examples 1, 3 (pp. 842, 844)

Verify that each of the following is an identity.

1. $\tan \theta (\cot \theta + \tan \theta) = \sec^2 \theta$

2. $\tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta$

3. $\frac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$

4. $\frac{1 + \tan^2 \theta}{\csc^2 \theta} = \tan^2 \theta$

5. $\frac{\sin \theta}{\sec \theta} = \frac{1}{\tan \theta + \cot \theta}$

6. $\frac{\sec \theta + 1}{\tan \theta} = \frac{\tan \theta}{\sec \theta - 1}$

Example 2 (p. 843)

7. STANDARDIZED TEST PRACTICE Which expression can be used to form an identity with $\frac{\sec \theta + \csc \theta}{1 + \tan \theta}$?

A $\sin \theta$ **B** $\cos \theta$ **C** $\tan \theta$ **D** $\csc \theta$

Exercises

HOMEWORK HELP	
For Exercises	See Examples
8–21	1–3

Verify that each of the following is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$

9. $\cot \theta (\cot \theta + \tan^2 \theta) = \csc^2 \theta$

10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$

11. $\sin \theta \sec \theta \cot \theta = 1$

12. $\frac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$

13. $\frac{1 - 2\cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$

14. $\cot \theta \csc \theta = \frac{\cot \theta + \csc \theta}{\sin \theta + \tan \theta}$

15. $\sin \theta + \cos \theta = \frac{1 + \tan \theta}{\sec \theta}$

16. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$

17. $\frac{\sin \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta$

18. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity.

19. Show that $1 + \cos \theta$ and $\frac{\sin^2 \theta}{1 - \cos \theta}$ form an identity.

Real-World Link

Model rocketry was developed during the "space-race" era. The rockets are constructed of cardboard, plastic, and balsa wood, and are fueled by single-use rocket motors.

PHYSICS **For Exercises 20 and 21, use the following information.**
If an object is propelled from ground level, the maximum height that it reaches is given by $h = \frac{v^2 \sin^2 \theta}{2g}$, where θ is the angle between the ground and the initial path of the object, v is the object's initial velocity, and g is the acceleration due to gravity, 9.8 meters per second squared.

20. Verify the identity $\frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta}$.

21. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80° with the ground. Find the maximum height of the rocket.

Verify that each of the following is an identity.

22. $\frac{1 + \sin \theta}{\sin \theta} = \frac{\cot^2 \theta}{\csc \theta - 1}$

23. $\frac{1 + \tan \theta}{1 + \cot \theta} = \frac{\sin \theta}{\cos \theta}$

24. $\frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1$

25. $1 + \frac{1}{\cos \theta} = \frac{\tan^2 \theta}{\sec \theta - 1}$

26. $1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta$

27. $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$

28. $\frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$

29. $\frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta$

Graphing Calculator

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation *might* be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following *may be* or *is not* an identity.

30. $\cot x + \tan x = \csc x \cot x$

31. $\sec^2 x - 1 = \sin^2 x \sec^2 x$

32. $(1 + \sin x)(1 - \sin x) = \cos^2 x$

33. $\frac{1}{\sec x \tan x} = \csc x - \sin x$

34. $\frac{\sec^2 x}{\tan x} = \sec x \csc x$

35. $\frac{1}{\sec x} + \frac{1}{\csc x} = 1$

EXTRA PRACTICE
See pages 923, 939.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. **OPEN ENDED** Write a trigonometric equation that is *not* an identity. Explain how you know it is not an identity.

37. **Which One Doesn't Belong?** Identify the equation that does not belong with the other three. Explain your reasoning.

$\sin^2 \theta + \cos^2 \theta = 1$ $\qquad$ $1 + \cot^2 \theta = \csc^2 \theta$

$\sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta$ $\qquad$ $\tan^2 \theta + 1 = \sec^2 \theta$

38. **CHALLENGE** Present a logical argument for why the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ is true when $0 \le x \le 1$.

39. ***Writing in Math*** Use the information on pages 842 and 843 to explain why you cannot perform operations to each side of an unverified identity and explain why you cannot use the graphs of two expressions to verify an identity.

STANDARDIZED TEST PRACTICE

40. ACT/SAT Which of the following is not equivalent to $\cos \theta$?

A $\dfrac{\cos \theta}{\cos^2 \theta + \sin^2 \theta}$

B $\dfrac{1 - \sin^2 \theta}{\cos \theta}$

C $\cot \theta \sin \theta$

D $\tan \theta \csc \theta$

41. REVIEW Which of the following is equivalent to $\sin \theta + \cot \theta \cos \theta$?

F $2 \sin \theta$

G $\dfrac{1}{\sin \theta}$

H $\cos^2 \theta$

J $\dfrac{\sin \theta + \cos \theta}{\sin^2 \theta}$

Spiral Review

Find the value of each expression. (Lesson 14-3)

42. $\sec \theta$, if $\tan \theta = \frac{1}{2}$; $0° < \theta < 90°$

43. $\cos \theta$, if $\sin \theta = -\frac{2}{3}$; $180° < \theta < 270°$

44. $\csc \theta$, if $\cot \theta = -\frac{7}{12}$; $90° < \theta < 180°$

45. $\sin \theta$, if $\cos \theta = \frac{3}{4}$; $270° < \theta < 360°$

State the amplitude, period, and phase shift of each function. Then graph each function. (Lesson 14-2)

46. $y = \cos (\theta - 30°)$

47. $y = \sin (\theta - 45°)$

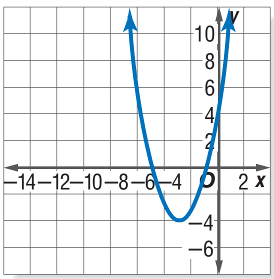
48. $y = 3 \cos \left(\theta + \frac{\pi}{2}\right)$

49. COMMUNICATIONS The carrier wave for a certain FM radio station can be modeled by the equation $y = A \sin (10^7 \cdot 2\pi t)$, where A is the amplitude of the wave and t is the time in seconds. Determine the period of the carrier wave. (Lesson 14-1)

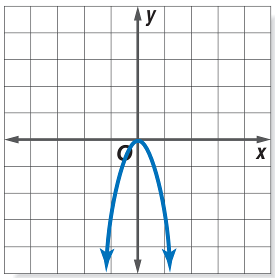
50. BUSINESS A company estimates that it costs $0.03x^2 + 4x + 1000$ dollars to produce x units of a product. Find an expression for the average cost per unit. (Lesson 6-3)

Use the related graph of each equation to determine its solutions. (Lesson 5-2)

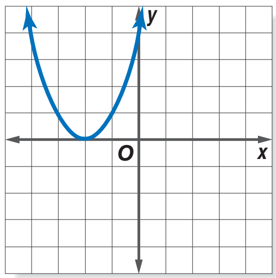
51. $y = x^2 + 6x + 5$



52. $y = -3x^2$



53. $y = x^2 + 4x - 4$



GET READY for the Next Lesson

PREREQUISITE SKILL **Simplify each expression.** (Lessons 7-5)

54. $\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

55. $\frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

56. $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2}$

57. $\frac{1}{2} - \frac{\sqrt{3}}{4}$